Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

Part I: Tuesday, July 10, 2018, 12:00pm - 4:00pm
Proctors: Mak Paranjape and Ed Van Keuren

Instructions:

- **Please put your assigned number on the first page of every problem!**
- This is a closed-book, closed-notes exam. The only electronic devices allowed are calculators provided by the department.
- Each problem is worth 50 points.
- You should submit work for all of the problems. In many cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- Please write your solution for each problem on separate sheets of paper.
- Show all your work.

Electricity and Magnetism
Comprehensive Exam Question

Note: The percentage of the grade allocated for each question in this exam has been explicitly stated.

Q1. To ask a physicist to state the value of the acceleration due to gravity, g, is trivial. So too is knowing the value of the AC voltage from a common North American household electrical outlet. When answering the following questions, be **absolutely precise** with your result, and express your answer with only **2 significant figures**:

a. State the value of "g".

g=__________________________ **(4%)**

b. i) Clearly and unambiguously state the numerical value of the electrical outlet voltage.

v=______________(zero-to-peak/peak-to-peak/effective value/average value?) **(8%)**

ii) Since electrical outlets are AC, what is the frequency of the voltage?

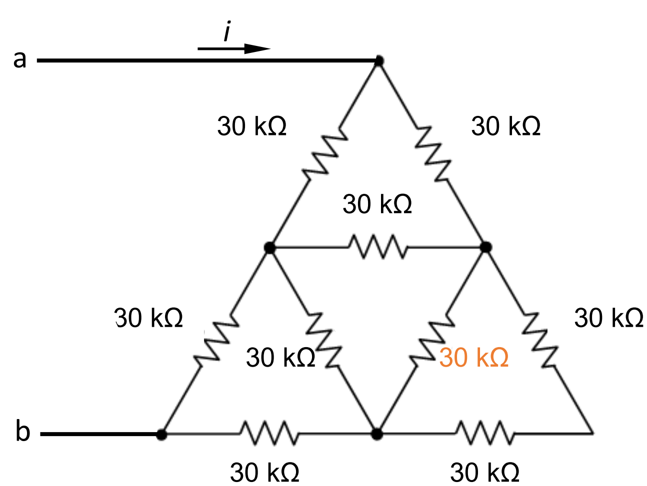
f=___________________________ **(8%)**

Q2. For the following resistive network, you are to determine the equivalent resistance between points a-b and find the current, *i* (shown), assuming the network is connected to a 17 v_{AC} source. **But**, you have a choice in the way you will solve the problem! You can either solve it:

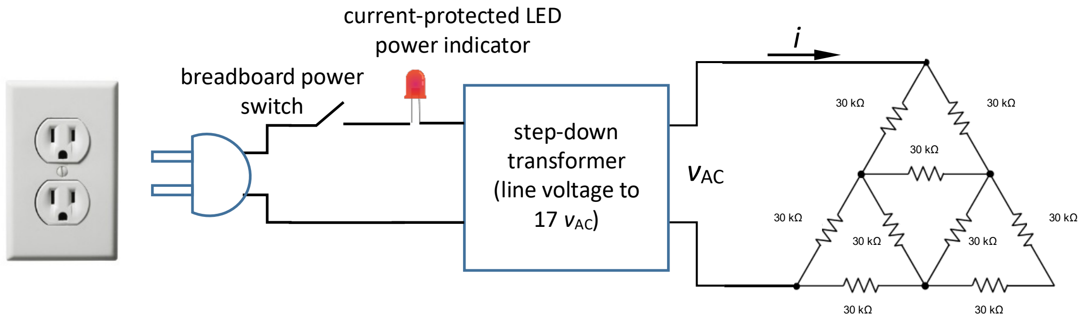
1. using theoretical calculations of series and parallel resistors (**Hint** - you may want to consider replacing the orange-colored 30 kΩ resistor with two 15 kΩ series resistors. This may seem like a pointless exercise, but critically examine the *form* of the resulting network),

or,

2. by physically performing an experimental measurement, since the resistive network has been physically wired up and connected to an AC voltage source (17 v_{AC}) using a simple breadboard system (the blue box), described later.



The (blue) breadboard circuit layout is as follows:



Using the provided digital multi-meter (DMM), measure the value for current, *i*, shown in the figure. **Measure** the equivalent resistance R_{ab} seen by the 17 v_{AC} source by using the resistance setting on the DMM.

Your grade will be determined based on **only one** solution method that you explicitly select to be graded. Of course, you may elect to do both methods as a means to confirm your answers, however, only one method will be graded. Select which method you would like graded:

☐ theoretical

☐ experimental

i= ____________________________ **(25**%)

R_{ab} = ____________________________ **(25**%)

Q3. Derive the wave equations for both electric field and magnetic field for an electromagnetic wave propagating through conductive matter, where typically, ρ=0 but J $\neq 0$. The following vector identities may (or may not) be useful:

$$\nabla \cdot (\nabla \times F) = 0$$

$$\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$$

$$\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$$

(**30%**)

Physics 2018 Comprehensive Exam: Electricity and Magnetism

Consider a raindrop of radius R that is suspended in air. Assume that a fraction of the water molecules in the drop are charged, i.e., one electron was removed from each of these molecules and they became electrically charged. These charged molecules will migrate to the surface of the drop and form a shell of charge Q_0 of radius R. Assume that $|Q_0|$ is much larger than $|q_e|$, where q_e is the charge of a single electron.

1. What is the electrostatic potential, $V(r)$, of the drop for $r > R$?
2. The work done to remove the electrons and charge the drop is equal and opposite to the work needed to bring all the electrons back and neutralize the drop. Calculate the work done to charge the drop using the following steps:
   a. Calculate the work dW needed to add a charge dq to the charged drop (assume that the additional charge dq is brought from $r = \infty$).
   b. Calculate the total work W needed to add a total charge $-Q_0$ and neutralize the charged drop.
3. Assume that the radius of the drop is 1 mm and the fraction of charged water molecules is 1%. Calculate Q_0 and the work done to charge the drop.

You will need the following parameters:
$\frac{1}{4\pi\epsilon_0} = 9 \cdot 10^9 \text{ Nm}^2/\text{C}^{-2}$

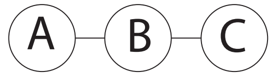
$q_e = 1.6 \cdot 10^{-19}$ C

Avogadro's number: $N_A = 6.02 \text{ x } 10^{23}$

Density of water: 1000 kg / m^3

Atomic mass of oxygen is 16 (One mole of oxygen weighs 0.016 kg)

1. Consider an electron of a linear triatomic molecule formed by three equidistant atoms (A, B, C) as shown below:



Use $|\phi_A\rangle$, $|\phi_B\rangle$, and $|\phi_C\rangle$ to denote three orthonormal states of the electron, corresponding to three wavefunctions localized, respectively, about the nuclei of atoms A, B, and C.

When the possibility of the electron jumping from one nucleus to another is neglected, the energy (of the electron) is described by the Hamiltonian $\hat{H}_0$, whose eigenstates are $|\phi_A\rangle$, $|\phi_B\rangle$, and $|\phi_C\rangle$ with the same eigenvalue E_0.

The couplings between the nuclei are described by $\hat{W}$, an additional contribution to the Hamiltonian defined by:

$$\begin{aligned} \hat{W}|\phi_A\rangle &= -a|\phi_B\rangle \\ \hat{W}|\phi_B\rangle &= -a|\phi_A\rangle - a|\phi_C\rangle \\ \hat{W}|\phi_C\rangle &= -a|\phi_B\rangle, \end{aligned}$$

where a is a real, positive constant.

(a) Write the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$ in the $\{|\phi_A\rangle, |\phi_B\rangle, |\phi_C\rangle\}$ representation.

(b) Calculate the energies and stationary states of $\hat{H}$.

(c) At time $t = 0$, a measurement is made and the electron is found to be localized on nucleus A.

   i. Determine the state of the system at an arbitrary time $t \geq 0$.

   ii. Let $\hat{D}$ be the observable whose eigenstates are $|\phi_A\rangle$, $|\phi_B\rangle$, and $|\phi_C\rangle$ with corresponding eigenvalues $-d$, 0, and d. $\hat{D}$ is measured at time $t \geq 0$. What values can be found and with what probabilities?

   iii. Are there times when the electron is localized perfectly on atom B?

   iv. Are there times when the electron is localized perfectly on atom C?

(d) When the initial state of the electron is arbitrary, list all non-zero frequencies that could appear in a Fourier decomposition of the time-evolution of the expectation value of a generic, time-independent operator $\hat{O}$.

(e) When the initial state of the electron is arbitrary, what non-zero frequencies can appear in a Fourier decomposition of the evolution of $\langle\hat{D}\rangle(t)$, where $\hat{D}$ is the operator defined above? (Be sure the justify which ones are included and excluded.)

Comprehensive exam question 2018
Quantum Mechanics

Consider a *two-dimensional isotropic simple harmonic oscillator* given by the following Hamiltonian:

$$\mathcal{H} = \hbar\omega(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1). \tag{1}$$

The raising and lowering operators satisfy the usual commutation relations, with the only nonzero commutators being

$$[\hat{a}_1, \hat{a}_1^\dagger] = 1 \quad \text{and} \quad [\hat{a}_2, \hat{a}_2^\dagger] = 1. \tag{2}$$

All other commutators vanish (including those between any 1 and 2 operators).
The (normalized) energy eigenstates are given by

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!}} \frac{1}{\sqrt{n_2!}} \left(\hat{a}_1^\dagger\right)^{n_1} |0\rangle_1 \otimes \left(a_2^\dagger\right)^{n_2} |0\rangle_2, \tag{3}$$

where $\hat{a}_i|0\rangle = 0$ for $i = 1, 2$.
In the 1950s, Julian Schwinger devised a way to use these operators to represent angular momentum states. You will explore this mapping in this problem.

(a) *(9 points)* Define the operators $\hat{J}_+ = \hbar\hat{a}_1^\dagger \hat{a}_2$, $\hat{J}_- = \hbar\hat{a}_2^\dagger \hat{a}_1$, and $\hat{J}_z = \hbar(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)/2$. Verify the angular momentum commutator algebra given by

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z, \quad [\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+, \quad \text{and} \quad [\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_-. \tag{4}$$

(b) *(6 points)* Define $\hat{J}_x = (\hat{J}_+ + \hat{J}_-)/2$ and $\hat{J}_y = (\hat{J}_+ - \hat{J}_-)/(2i)$ and compute the commutator $[\hat{J}_x, \hat{J}_y] =?$. Be sure to express your answer in terms of the Cartesian $\hat{J}$ operators and show all of your work. You may use commutators calculated in (a) in your derivation, if desired.

(c) *(10 points)* Consider the state $|j, m = j\rangle$ in the angular momentum representation that satisfies $\hat{J}_+|j, m = j\rangle = 0$. Use this condition on the set of states $\{|n_1, n_2\rangle\}$, for all nonnegative integers n_1 and n_2, to determine what states are $|j, m = j\rangle$ states. Next, by using your knowledge about the $|j, m\rangle$ multiplet, determine formulas for j and m in terms of n_1 and n_2. *Hint*: you should be able to verify that the states $|n_1, n_2\rangle$ are already eigenstates of $\hat{J}_z$; use the properties of the raising and lowering operators acting on these states to infer the relationship between the quantum numbers n_1 and n_2 and j and m.

(d) *(6 points)* Using $\hat{J}^2 = (\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)/2 + \hat{J}_z^2$, compute $\hat{J}^2|j, m\rangle = \hat{J}^2|n_1, n_2\rangle$, for the values of n_1 and n_2 which give the general $|j, m\rangle$ state that you found in part (c). You must show all of the algebra, perform the algebra with the simple harmonic oscillator raising and lowering operators, and express your final answer for the eigenvalue as a function of j and m. Do not just quote the known answer.

(e) *(15 points)* We now work with the multiplet given by all states of the form $(\hat{J}_+)^\alpha|n_1 = 0, n_2 = 2\rangle$ and α any of the allowed nonnegative integers consistent with your results in part (c). We define the angular momentum matrices via

$$\left(\hat{J}_z\right)_{m,m'} = \langle j, m|\hat{J}_z|j, m'\rangle, \quad \left(\hat{J}_+\right)_{m,m'} = \langle j, m|\hat{J}_+|j, m'\rangle, \quad \text{and} \quad \left(\hat{J}_-\right)_{m,m'} = \langle j, m|\hat{J}_-|j, m'\rangle. \tag{5}$$

Determine the three angular momentum matrices for the representation created from the $|n_1 = 0, n_2 = 2\rangle$ state. Use the simple harmonic oscillator algebra and recall (or rederive) what $\hat{a}$ and $\hat{a}^\dagger$ do on simple harmonic oscillator number states.

(f) *(4 points)* Construct the matrices for $\hat{J}_x$ and $\hat{J}_y$, using the results from part (e) and the relationship between the raising and lowering operators and the Cartesian operators.

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Part II: Thursday, July 12, 2018, 12:00pm - 4:00pm
Proctors: Ed Van Keuren and Amy Liu

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